ON THE DETERMINATION OF THE POSITION OF A NONLINEAR SAMPLED DATA SYSTEM IN PHASE SPACE

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In many controlled systems, in order to determine the control algorithm, it is necessary to have some information on the position of the controlled system in the phase space. Such information cannot always be obtained since some of the phase coordinates of the system may not be accessible for measurement, and furthermore, information about the position of the system of orientation relative to which the position of the controlled system is to be determined, might be missing.

In [1,2] one of the possible indirect methods for obtaining some information about the position of linear and nonlinear continuous systems in phase space was considered. Below the extension of this method is given for sampled data [impulsive] controlled systems.

1. We shall proceed from the following system of difference equations, describing the motion of the controlled sampled data system

$$\sum_{k=1}^{n} f_{jk}(T)y_k = x_j(t) + \psi_j(y_1, Ty_1, \ldots, T^{m_1-1}y_1, \ldots, y_n, Ty_n, \ldots, T^{m_n-1}y_n, t) \qquad (j = 1, \ldots, n) \tag{1.1}$$

where y_k are the generalized coordinates of the system,, and $x_j(t)$ are the given external forces. By $f_{jk}(T)$ are represented polynomials of T, the coefficients of which are given functions of time, and T appears as the lead operator determined by the relation

$$T^{\mu} y_k = y_k(t + \mu\tau) \tag{1.2}$$

where τ is some constant value. The highest degree of T in the polynomials $f_{jk}(T)$ $(j = 1, \ldots, n)$ for a given k, is represented by m_k.

The functions $\psi_j (j = 1, \ldots, n)$ on the right-hand side of equations (1.1) appear as some nonlinear functions of their arguments. These functions are assumed to be continuous with respect to all their arguments in some closed domain, and satisfy in that domain the conditions of Lipschitz relative to the arguments

$$y_1, Ty_1, \ldots, T^{m_1-1} y_1, \ldots, y_n, Ty_n, \ldots, T^{m_n-1} y_n$$

Equations (1.1) can be brought [3] to the following form*

$$Tz_\nu + \sum_{k=1}^{r} a_{\nu k}(t) z_k = X_\nu(t) + \Psi_\nu(z_1, \ldots, z_r, t) \qquad (\nu = 1, \ldots, r) \tag{1.3}$$

where

$$z_1 = y_1, \quad z_2 = Ty_1, \ldots, \quad z_{m_1} = T^{m_1-1} y_1, \ldots, \quad z_r = T^{m_n-1} y_n \tag{1.4}$$

$$r = m_1 + m_2 + \ldots + m_n \tag{1.5}$$

$$X_{\sigma_j}(t) = \sum_{k=1}^{n} \frac{B_{kj}(t)}{\Delta^*(t)} x_k(t)$$

$$\Psi_{\sigma_j}(z_1, \ldots, z_r, t) = \sum_{k=1}^{n} \frac{B_{kj}(t)}{\Delta^*(t)} \psi_k(z_1, \ldots, z_r, t) \qquad (\sigma_j = \sigma_1, \ldots, \sigma_n) \tag{1.6}$$

$$\sigma_1 = m_1, \qquad \sigma_2 = m_1 + m_2, \ldots, \sigma_n = r \tag{1.7}$$

and the functions $X_\mu(t)$, $\Psi_\mu(z_1, \ldots, z_r, t)$, for which $\mu \neq \sigma_l \, (l = 1, \ldots, n)$, are identically equal to zero.

Let us represent by $\Delta^*(t)$ in expression (1.6) the determinant of the coefficients $b_{jk}(t)$ with which the quantities $T^{m_k} y_k(t)$ enter the left-hand side of equations (1.1)

$$\Delta^*(t) = | b_{jk}(t) | \tag{1.8}$$

whereupon it is assumed that the determinant is not identically equal to zero. The B_{kj} represent the minors of the elements b_{kj} in the determinant (1.8).

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* I take advantage of the opportunity to remark that in [3] a printing mistake was found. On page 930 the formula (1.34) must be read:

$$K_{s_i}(j_1\tau) = \sum_{\xi=1}^{m} A_{p_\xi s_i}(j_1\tau) [r_{p_\xi} - g_{p_\xi}(j_1\tau)] \quad (i = 1, \ldots, m)$$

Introducing the matrices

$$z\ (t)=\|z_\nu(t)\|, \quad a\ (t)=\|a_{\nu k}\ (t)\|, \qquad X\ (t)=\|X_\nu\ (t)\|$$
$$\Psi(z_1(t), \ldots, z_r\ (t), t)=\|\Psi_\nu\ (z_1\ (t), \ldots, z_r\ (t),\ t)\| \tag{1.9}$$

it is possible to replace the system of scalar equations (1.3) by the matrix equation

$$z\ (t+\tau)+a\ (t)\ z\ (t)=X\ (t)\ +\Psi\ (z_1\ (t), \ldots, z_r\ (t),\ t) \tag{1.10}$$

In order to solve the difference equation (1.10) it is also indispensable to give the matrix

$$z^*\ (t)=\|z_\nu^*\ (t)\| \qquad (0<t<\tau) \tag{1.11}$$

defined on the interval of time $0 < t < \tau$ by the law of variation of the sought for functions $z_\nu(t)$ ($\nu = 1, \ldots, r$) on this (initial) interval.

Denoting by $\theta(t)$ the fundamental matrix for the homogeneous linear matrix equation

$$z\ (t+\tau)+a\ (t)\ z\ (t)=0 \tag{1.12}$$

and introducing the function

$$N\,(t,\,j\tau)=\theta\,(t)\,\theta^{-1}\,(t-\vartheta\tau+j\tau) \tag{1.13}$$

which represents the matrix weighing function for the matrix difference equation (1.12), (the inverse matrix is represented by $\theta^{-1}(t)$), it is possible to pass [3] from the system of nonlinear difference equations (1.3) to the system of nonlinear equations in finite sums

$$z_\nu(t)=\sum_{k=1}^{r} N_{\nu k}(t,0)\,z_k^*(t-\vartheta\tau)+\sum_{i=1}^{n}\sum_{j=1}^{\vartheta} N_{\nu\sigma_i}(t,j\tau)\,X_{\sigma_i}(t-\vartheta\tau+j\tau-\tau)+$$
$$+\sum_{i=1}^{n}\sum_{j=1}^{\vartheta} N_{\nu\sigma_i}(t,j\tau)\,\Psi_{\sigma_i}(z_1(t-\vartheta\tau+j\tau-\tau),\ldots,z_r(t-\vartheta\tau+j\tau-\tau),\ t-\vartheta\tau+j\tau-\tau)$$
$$(\vartheta=[t/\tau]) \qquad (\nu=1,\ldots,r) \tag{1.14}$$

where ϑ represents the fraction t/τ.

Equations (1.14) are equivalent to the combination of the difference equations (1.3) and the given law of variation of the sought functions $z_\nu(t)$ ($\nu = 1, \ldots, r$) on the initial interval of time $0 < t < \tau$. They are analogous to the integral equations in the continuous analysis.

Substituting for X_{σ_i} and Ψ_{σ_i} their expressions (1.6) and denoting

$$W_{\nu l}(t, j\tau) = \sum_{i=1}^{n} N_{\nu\sigma_i}(t, j\tau) \frac{B_{li}(t - \vartheta\tau + j\tau - \tau)}{\Delta^*(t - \vartheta\tau + j\tau - \tau)} \qquad \begin{pmatrix} \nu = 1, \ldots, r \\ l = 1, \ldots, n \end{pmatrix} \tag{1.15}$$

it is possible to write the equations (1.14) in the following form

$$z_\nu(t) = \sum_{k=1}^{r} N_{\nu k}(t, 0) z_k^*(t - \vartheta\tau) + \sum_{l=1}^{n} \sum_{j=1}^{\vartheta} W_{\nu l}(t, j\tau) x_l(t - \vartheta\tau + j\tau - \tau) +$$
$$+ \sum_{l=1}^{n} \sum_{j=1}^{\vartheta} W_{\nu l}(t, j\tau) \psi_l(z_1(t - \vartheta\tau + j\tau - \tau), \ldots, z_r(t - \vartheta\tau + j\tau - \tau), \quad t - \vartheta\tau + j\tau - \tau)$$
$$(\nu = 1, \ldots, r) \tag{1.16}$$

2. The solution of the system of equations (1.16) can be found if one knows the law of variation of the sought functions $z_\nu(t)(\nu = 1, \ldots, r)$ on the interval of time $0 < t < \tau$, i.e. if the matrix (1.11) is known.

Nevertheless, in many problems the law of variation of the sought functions $z_\nu(t)(\nu = 1, \ldots, r)$ on the initial interval of time $0 < t < \tau$ is not known, and only one of the phase coordinates z_s can be measured, whereupon the position of the beginning of the measurement of that coordinate is also unknown. The law, according to which the external forces $x_l(t)(l = 1, \ldots, n)$ change, is assumed to be known.

For the determination of the sought functions $z_\nu(t)(\nu = 1, \ldots, r)$ under the conditions which were assumed here, we shall choose a new arbitrary origin for the phase coordinate z_s and shall let it remain constant. Since the phase coordinate z_s is accessible for measurement, then with these measurements it is possible to find a law, according to which the variations of the phase coordinate z_s

$$S(\gamma_i\tau + \varepsilon) \qquad 0 < \varepsilon < \tau \qquad (i = 1, \ldots, r + 1)$$

vary with respect to the new origin on the interval of time $\gamma_i\tau < t < (\gamma_i + 1)\tau$, where $\gamma_i(i = 1, \ldots, r + 1)$ are some integers. Since

$$S(\gamma_i\tau + \varepsilon) = S^* + z_s(\gamma_i\tau + \varepsilon) \qquad (i = 1, \ldots, r + 1) \tag{2.1}$$

where S^* is the variation of the new origin with respect to the initial one, then denoting

$$S(\gamma_{\mu+1}\tau + \varepsilon) - S(\gamma_\mu\tau + \varepsilon) = L_\mu(\varepsilon), \qquad 0 < \varepsilon < \tau \qquad (\mu = 1, \ldots, r) \tag{2.2}$$

we shall get the following relations (which do not contain the unknown quantity S^*) between the increments of the phase coordinate z_s and the results of measurements

$$z_s(\gamma_{\mu+1}\tau+\varepsilon)-z_s(\gamma_\mu\tau+\varepsilon)=L_\mu(\varepsilon)\qquad 0<\varepsilon<\tau\qquad (\mu=1,\ldots,r)\tag{2.3}$$

Using (1.16), the expressions (2.3) can be brought to the following form

$$\begin{aligned}\sum_{k=1}^{r}c_{\mu k}(\varepsilon)\,z_k^*(\varepsilon)=P_\mu(\varepsilon)-\qquad\qquad(0<\varepsilon<\tau;\ \mu=1,\ldots,r)\\ -\sum_{l=1}^{n}\sum_{j=1}^{\gamma_{\mu+1}}W_{sl}(\gamma_{\mu+1}\tau+\varepsilon,\ j\tau)\,\psi_l(z_1(\varepsilon+j\tau-\tau),\ldots,z_r(\varepsilon+j\tau-\tau),\ \varepsilon+j\tau-\tau)+\\ +\sum_{l=1}^{n}\sum_{j=1}^{\gamma_\mu}W_{sl}(\gamma_\mu\tau+\varepsilon,\ j\tau)\,\psi_l(z_1(\varepsilon+j\tau-\tau),\ldots,z_r(\varepsilon+j\tau-\tau),\ \varepsilon+j\tau-\tau)\end{aligned}\tag{2.4}$$

where

$$c_{\mu k}(\varepsilon)=N_{sk}(\gamma_{\mu+1}\tau+\varepsilon,0)-N_{sk}(\gamma_\mu\tau+\varepsilon,0),\ \ 0<\varepsilon<\tau,\ (\mu,k=1,\ldots,r)\tag{2.5}$$

$$\begin{aligned}P_\mu(\varepsilon)=L_\mu(\varepsilon)-\sum_{l=1}^{n}\sum_{j=1}^{\gamma_{\mu+1}}W_{sl}(\gamma_{\mu+1}\tau+\varepsilon,\ j\tau)\,x_l(\varepsilon+j\tau-\tau)+\\ +\sum_{l=1}^{n}\sum_{j=1}^{\gamma_\mu}W_{sl}(\gamma_\mu\tau+\varepsilon,\ j\tau)\,x_l(\varepsilon+j\tau-\tau),\qquad 0<\varepsilon<\tau,\qquad(\mu=1,\ldots,r)\end{aligned}\tag{2.6}$$

From equations (2.4), it follows that

$$\begin{aligned}z_i^*(\varepsilon)=\frac{1}{\Lambda(\varepsilon)}\sum_{\mu=1}^{r}A_{\mu i}(\varepsilon)\,P_\mu(\varepsilon)-\frac{1}{\Lambda(\varepsilon)}\sum_{\mu=1}^{r}A_{\mu i}(\varepsilon)\times\\ \times\Big[\sum_{l=1}^{n}\sum_{j=1}^{\gamma_{\mu+1}}W_{sl}(\gamma_{\mu+1}\tau+\varepsilon,\ j\tau)\ \psi_l(z_1(\varepsilon+j\tau-\tau),\ldots,z_r(\varepsilon+j\tau-\tau),\ \varepsilon+j\tau-\tau)-\\ -\sum_{l=1}^{n}\sum_{j=1}^{\gamma_\mu}W_{sl}(\gamma_\mu\tau+\varepsilon,\ j\tau)\ \psi_l(z_1(\varepsilon+j\tau-\tau),\ldots,z_r(\varepsilon+j\tau-\tau),\ \varepsilon+j\tau-\tau)\Big]\\ 0<\varepsilon<\tau\qquad (i=1,\ldots,r)\end{aligned}\tag{2.7}$$

where

$$\Lambda(\varepsilon)=\begin{vmatrix}c_{11}(\varepsilon) & c_{12}(\varepsilon)\ .\ .\ . & c_{1r}(\varepsilon)\\ . & . \ . \ . & .\\ c_{r1}(\varepsilon) & c_{r2}(\varepsilon)\ .\ .\ . & c_{rr}(\varepsilon)\end{vmatrix}\tag{2.8}$$

and $A_{\mu i}(\mu,\ i=1,\ \ldots,\ r)$ are the minors of the elements $c_{\mu i}$ in the determinant (2.8).

Since $\vartheta = [t/\tau]$, then $0 < t - \vartheta\tau < \tau$, and therefore the functions $z_k^*(t - \vartheta\tau)(k = 1, \ldots, r)$ entering equations (1.16) can be replaced by the expressions (2.7) found for $z_k^*(\varepsilon)(0 < \varepsilon < \tau)$. Thus, we shall get the following system of nonlinear equations (which do not contain the unknown function $z_k^*(\epsilon)$) in finite sums, with respect to the sought functions $z_i(t)(i = 1, \ldots, r)$

$$\begin{gathered}
z_\nu(t) = G_\nu(t) - \sum_{\mu=1}^{r} V_{\nu\mu}(t)\Big[\sum_{l=1}^{n}\sum_{j=1}^{\gamma_{\mu+1}} W_{sl}(t - \vartheta\tau + \gamma_{\mu+1}\tau, j\tau)\times \\
\times \psi_l(z_1(t - \vartheta\tau + j\tau - \tau), \ldots, z_r(t - \vartheta\tau + j\tau - \tau), t - \vartheta\tau + j\tau - \tau) - \\
- \sum_{l=1}^{n}\sum_{j=1}^{\gamma_\mu} W_{sl}(t - \vartheta\tau + \gamma_\mu\tau, j\tau)\times \\
\times \psi_l(z_1(t - \vartheta\tau + j\tau - \tau), \ldots, z_r(t - \vartheta\tau + j\tau - \tau), t - \vartheta\tau + j\tau - \tau)\Big] + \\
+ \sum_{l=1}^{n}\sum_{j=1}^{\vartheta} W_{\nu l}(t, j\tau)\psi_l(z_1(t - \vartheta\tau + j\tau - \tau), \ldots, z_r(t - \vartheta\tau + j\tau - \tau), t - \vartheta\tau + j\tau - \tau) \\
(\nu = 1, \ldots, r)
\end{gathered} \tag{2.9}$$

where

$$G_\nu(t) = \frac{1}{\Lambda(t - \vartheta\tau)}\sum_{\xi=1}^{r}\sum_{\mu=1}^{r} N_{\nu\xi}(t, 0)A_{\mu\xi}(t - \vartheta\tau)P_\mu(t - \vartheta\tau) + $$

$$+ \sum_{l=1}^{n}\sum_{j=1}^{\vartheta} W_{\nu l}(t, j\tau)x_l(t - \vartheta\tau + j\tau - \tau) \qquad (\nu = 1, \ldots, r) \tag{2.10}$$

$$V_{\nu\mu}(t) = \frac{1}{\Lambda(t - \vartheta\tau)}\sum_{\xi=1}^{r} N_{\nu\xi}(t, 0)A_{\mu\xi}(t - \vartheta\tau) \qquad (\nu, \mu = 1, \ldots, r) \tag{2.11}$$

For discrete values of the argument $t = \vartheta\tau(\vartheta = 1, 2, \ldots)$ equations (2.9) take the form

$$\begin{gathered}
z_\nu(\vartheta\tau) = G_\nu(\vartheta\tau) - \sum_{\mu=1}^{r} V_{\nu\mu}(\vartheta\tau)\Big[\sum_{l=1}^{n}\sum_{j=1}^{\gamma_{\mu+1}} W_{sl}(\gamma_{\mu+1}\tau, j\tau)\times \\
\times \psi_l(z_1(j\tau - \tau), \ldots, z_r(j\tau - \tau), j\tau - \tau) - \\
- \sum_{l=1}^{n}\sum_{j=1}^{\gamma_\mu} W_{sl}(\gamma_\mu\tau, j\tau)\psi_l(z_1(j\tau - \tau), \ldots, z_r(j\tau - \tau), j\tau - \tau)\Big] + \\
+ \sum_{l=1}^{n}\sum_{j=1}^{\vartheta} W_{\nu l}(\vartheta\tau, j\tau)\psi_l(z_1(j\tau - \tau), \ldots, z_r(j\tau - \tau), j\tau - \tau)\Big] \\
(\nu = 1, \ldots, r)
\end{gathered} \tag{2.12}$$

In order to search for values of the sought functions $z_\nu(t)(\nu = 1, \ldots, r)$ at least at the discrete points $t = \vartheta\tau(\vartheta = 1, 2, \ldots)$, it is sufficient to find the solution of the system of equations (2.12). The determination of the function $z_\nu(t)$ $(\nu = 1, \ldots, r)$ for a continuous argument t requires the solution of the system of equations (2.9). In order to solve equations (2.9) and (2.12) it is indispensable to apply numerical methods [4].

We shall note that the conditions of solvability of the system of equations (2.9) in the general case are not yet completely known in their entirety. For some particular cases, and more specifically for linear systems, this question has been studied in [5], where the conditions of solvability of the corresponding equations are called conditions of controllability of the controlled system.

3. In the case in which the nonlinear functions $\psi_l(z_1, \ldots, z_r, t)$ $(l = 1, \ldots, n)$ do not depend on some phase coordinate z_ρ, the number of equations constituting the system (2.9) decreases. Thus, for instance, if in the nonlinear functions ψ_l $(l = 1, \ldots, n)$ enters as much as one phase coordinate z_k

$$\psi_l = \psi_l(z_k(t),\ t) \qquad (l = 1, \ldots, n) \tag{3.1}$$

then in accordance with (2.9) it will be necessary to solve the following nonlinear equation relative to the unknown function $z_k(t)$

$$\begin{aligned} z_k(t) = G_k(t) - \sum_{\mu=1}^{r} V_{k\mu}(t) \times \\ \times\Big[\sum_{l=1}^{n}\sum_{j=1}^{\gamma_{\mu+1}} W_{sl}(t-\vartheta\tau+\gamma_{\mu+1}\tau,\ j\tau)\,\psi_l(z_k(t-\vartheta\tau+j\tau-\tau),\ \ t-\vartheta\tau+j\tau-\tau) - \\ -\sum_{l=1}^{n}\sum_{j=1}^{\gamma_\mu} W_{sl}(t-\vartheta\tau+\gamma_\mu\tau, j\tau)\,\psi_l(z_k(t-\vartheta\tau+j\tau-\tau), t-\vartheta\tau+j\tau-\tau)\Big] + \\ +\sum_{l=1}^{n}\sum_{j=1}^{\vartheta} W_{kl}(t,\ j\tau)\,\psi_l(z_k(t-\vartheta\tau+j\tau-\tau),\ t-\vartheta\tau+j\tau-\tau) \end{aligned} \tag{3.2}$$

The remaining phase coordinates z_ρ will be expressed in finite sums

$$\begin{aligned} z_\rho(t) = G_\rho(t) - \sum_{\mu=1}^{r} V_{\rho\mu}(t) \times \\ \times\Big[\sum_{l=1}^{n}\sum_{j=1}^{\gamma_{\mu+1}} W_{sl}(t-\vartheta\tau+\gamma_{\mu+1}\tau, j\tau)\,\psi_l(z_k(t-\vartheta\tau+j\tau-\tau),\ \ t-\vartheta\tau+j\tau-\tau) - \\ \sum_{l=1}^{n}\sum_{j=1}^{\gamma_\mu} W_{sl}(t-\vartheta\tau+\gamma_\mu\tau, j\tau)\,\psi_l(z_k(t-\vartheta\tau+j\tau-\tau), t-\vartheta\tau+j\tau-\tau)\Big] + \end{aligned}$$

$$+\sum_{l=1}^{n}\sum_{j=1}^{\vartheta} W_{\rho l}(t, j\tau)\,\psi_l(z_k(t-\vartheta\tau+j\tau-\tau),\quad t-\vartheta\tau+j\tau-\tau)$$

$$(\rho=1,\ldots,k-1,\ k+1,\ldots,r) \qquad (3.3)$$

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